

# CLASSIFICATION OF FRAMED LINKS IN 3-MANIFOLDS

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**ABSTRACT.** We present a short complete proof of the following Pontryagin theorem, whose original proof was complicated and has never been published in details: *Let  $M$  be a connected oriented closed smooth 3-manifold,  $L_1(M)$  be the set of framed links in  $M$  up to a framed cobordism, and  $\deg : L_1(M) \rightarrow H_1(M; \mathbb{Z})$  be the map taking a framed link to its homology class. Then for each  $\alpha \in H_1(M; \mathbb{Z})$  there is a 1-1 correspondence between the set  $\deg^{-1} \alpha$  and the group  $\mathbb{Z}_{2d(\alpha)}$ , where  $d(\alpha)$  is the divisibility of the projection of  $\alpha$  to the free part of  $H_1(M; \mathbb{Z})$ .*

## 1 Introduction

Throughout this paper let  $M$  be a connected oriented closed smooth 3-dimensional manifold. Denote by  $L_1(M)$  the set of 1-dimensional framed links in  $M$  up to a framed cobordism. The main purpose of this paper is to describe the set  $L_1(M)$ .

This classification problem appeared in Pontryagin's investigations connected with the calculation of the homotopy groups of spheres and, in a more general situation, of cohomotopy sets. *The cohomotopy set  $\pi^2(M) = [M; S^2]$*  is the set of continuous maps  $M \rightarrow S^2$  up to a homotopy. By the Pontryagin-Thom construction this set is in 1-1 correspondence with the set  $L_1(M)$ . Notice that the set of all nonzero vector fields on  $M$  up to homotopy, as well as the set of all oriented plane fields on  $M$  up to homotopy, is also in 1-1 correspondence with the set  $L_1(M)$ , because every orientable 3-manifold is parallelizable.

To state the main result we need the notions of *the natural orientation* on a framed link and *the degree* of a framed link, defined as follows. The link  $L$  is *naturally oriented* if for each point  $x \in L$  the tangent vector of the orientation together with the two vectors of the framing gives a positive basis of  $M$ . The *degree*  $\deg L$  of  $L$  is the homology class (with integral coefficients) of naturally oriented  $L$ . So we have a map

$$\deg : L_1(M) \rightarrow H_1(M; \mathbb{Z}).$$

The classical Hopf-Whitney theorem (1932-35) asserts that this map is always surjective.

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**Theorem 1 (L.S. Pontryagin).** *Let  $M^3$  be a connected oriented closed smooth 3-manifold. Then for each  $\alpha \in H_1(M^3; \mathbb{Z})$  there is a 1-1 correspondence between the sets  $\deg^{-1} \alpha$  and  $\mathbb{Z}_{2d(\alpha)}$ , where  $d(\alpha)$  is the divisibility of the projection of  $\alpha$  to the free part of  $H_1(M^3; \mathbb{Z})$ .*

**Example.** *The set of all maps  $f : S^1 \times S^1 \times S^1 \rightarrow S^2$  is up to homotopy in a bijective correspondence with the set of all 4-tuples  $(p, q, r, t)$ , where  $p, q, r \in \mathbb{Z}$  are the degrees of the restrictions of  $f$  to the 2-dimensional subtori,  $t \in \mathbb{Z}$ , for  $p = q = r = 0$ , and  $t \in \mathbb{Z}_{2\gcd(p, q, r)}$ , otherwise.*

Recall that the divisibility of zero is zero and the divisibility of a nonzero element  $\alpha \in G$  is  $\max\{d \in \mathbb{Z} \mid \exists \beta \in G : \alpha = d\beta\}$ . We denote  $\mathbb{Z}_d = \mathbb{Z}/d\mathbb{Z}$ , in particular,  $\mathbb{Z}_0 = \mathbb{Z}$ .

In this paper we give a short and direct proof of Theorem 1 (which was stated without proof in [Po]). In fact, Theorem 1 was not even properly stated in the paper [Po] itself (the paper was written in English), but only in the abstract (written in Russian) without any indication of the proof. The statement in the abstract is not clear, so we have borrowed it from [St].

The statement from [St] asserts that there is a 1-1 correspondence between

$$\deg^{-1} \alpha \quad \text{and} \quad \frac{\mathbb{Z}}{2\alpha \cap H_2(M; \mathbb{Z})},$$

which by the Poincaré duality, is equivalent to our statement of Theorem 1. There are reasons to believe that our proof is the same as the proof which Pontryagin had in mind, but never published it, going instead straight to the general case when  $M$  is an arbitrary polyhedron.

In the stable codimension  $n \geq 4$  there is an analogous theorem describing the set of framed links in  $n$ -manifolds [RSS]:

**Theorem 2 (L.S. Pontryagin, N. Steenrod, W.T. Wu).** *Let  $M$  be a connected oriented closed smooth  $n$ -manifold,  $n \geq 4$ . Then the degree map  $\deg : L_1(M) \rightarrow H_1(M; \mathbb{Z})$  is a bijection, if there exists  $\beta \in H_2(M; \mathbb{Z})$  such that  $\rho_2 \beta \cdot w_2(M) = 1 \pmod{2}$ . If such  $\beta$  does not exist, then  $\deg$  is a 2-1 map (that is, each  $\alpha \in H_1(M; \mathbb{Z})$  has exactly two preimages).*

Here  $w_2(M)$  is the Stiefel-Whitney class and  $\rho_2 : H_1(M; \mathbb{Z}) \rightarrow H_1(M; \mathbb{Z}_2)$  is the reduction modulo 2. In this paper we use an extension of the ideas of [RSS]. (Note that there is a misprint in the statement of this theorem in [RSS; Theorem 1a]).

The results stated above remain of sufficient interest up to the present – see [AK], [BP], [Du], [Go], [GG], and [Ka]. Note that Proposition 4.1 of [Go] is equivalent to our Theorem 1. However, even though [Go] gives the proof of this proposition, it is not written in details – in the notations of our paper (see §3 below), the statement (1) that the invariant  $h$  is well-defined and the surjectivity of  $h$  are indeed verified, whereas the proof of the *injectivity*, which is not evident, is absent from [Go].

Notice also that the statements of this result in [BP; Theorem 6.2.7] and [Du; Proposition 1] are erroneous, because of a different definition of the number  $d(\alpha)$ . (In these papers  $d(\alpha)$  is defined to be 0 if  $\alpha$  is a torsion element, otherwise it is the divisibility of  $\alpha$  in  $H_1(M^3; \mathbb{Z})$ . This is not equivalent to our definition.) An alternative approach to Theorem 1, by different methods, can be found in [AuKa]. A sketch of alternative proof can be found in [Ku; Proposition 2.1]. A more general result appears in [Ka; Theorem 2.4.7] and [GG; Proposition 7].

The plan of the paper is as follows: in §2 we first recall a nice geometric definition of the normal Euler class and we then prove another Pontryagin's classification theorem (see Remark after Lemma 3). In §3 we finally prove Theorem 1.

## 2 Preliminaries

We are going to use the following *geometric definition* of the normal Euler class, which is equivalent to other definitions. Let  $M^4$  be a closed oriented connected 4-manifold. Let  $L^2$  be a connected oriented manifold immersed in  $M^4$ . Let  $\nu(L)$  be the normal bundle of  $L$ . Identify  $L$  with the zero section of  $\nu(L)$ . Fix a natural orientation of  $\nu(L)$ . Take a general position section  $L'$  of  $\nu(L)$ . *The normal Euler class*  $\bar{e}(L) = \bar{e}(\nu(L)) \in \mathbb{Z}$  is the difference between the numbers of positive and negative intersection points of  $L$  and  $L'$ . Further denote by  $X \cap Y$  the difference between the numbers of positive and negative intersection points of  $X$  and  $Y$ .

We are also going to use the following *geometric definition* of the relative normal Euler class. Fix an orientation of  $M^3 \times [0; 1]$ . Let  $L_1 \subset M \times 1$  and  $L_2 \subset M \times 0$  be a pair of framed links, let  $L \subset M \times [0; 1]$  be a (unframed) cobordism between them. Fix a natural orientation of  $L$ , i.e. an orientation that induces natural orientations of  $L_1$  and  $L_2$ . Fix a natural orientation of  $\nu(L)$ . The first vector field of the framings of  $L_1$  and  $L_2$  can be considered as a section of  $\partial\nu(L)$ . Let  $L'$  be a general position extension of this section to a section of  $\nu(L)$ . *The relative normal Euler class*  $\bar{e}(L) \in \mathbb{Z}$  is the difference between the numbers of positive and negative intersection points of  $L$  and  $L'$ . If we reverse the orientation of  $M^3 \times [0; 1]$  (and, consequently, of  $L$ , because  $L$  is naturally oriented), then the sign of the integer  $\bar{e}(L)$  changes.

It can be shown that the class  $\bar{e}(L)$  is the complete obstruction to extension of the framing of  $\partial L$  to a framing of  $L$ .

**Lemma 3.** *Let  $L^2$  and  $M^4$  be a pair of connected oriented manifolds ( $M$  may have boundary). Suppose that  $L$  is immersed into  $M$ . Denote by  $[L] \in H_2(M; \mathbb{Z})$  the class of  $L$ . Denote by  $\sigma$  the difference between the numbers of positive and negative self-intersection points of  $L$ . Then*

$$\bar{e}(L) = [L] \cdot [L] - 2\sigma,$$

where we identify the group  $H_0(M; \mathbb{Z})$  with  $\mathbb{Z}$ . In particular, if  $M = N^3 \times I$  for some 3-manifold  $N^3$ , then  $\bar{e}(L) = -2\sigma$ .

**Remark.** In particular, this well-known lemma implies [RSS; Theorem 1.2b].

*Proof of Lemma 3.* Let  $\pi$  be the natural projection of a neighbourhood of  $L$  in  $\nu(L)$  to a small neighbourhood of  $L$  in  $M$ . Take a general position section  $L'$  of  $\nu(L)$  close to zero. The lemma now follows from

$$\bar{e}(L) = L \cap L' = \pi L \cap \pi L' - 2\sigma = [L] \cdot [L] - 2\sigma. \quad \square$$

## 3 Proof of Theorem 1

In order to construct a bijection  $h : \deg^{-1} \alpha \rightarrow \mathbb{Z}_{2d(\alpha)}$ , fix a framed circle  $L_1$  such that  $\deg L_1 = \alpha$  (clearly, such a circle exists). Take an arbitrary framed link  $L_2$  such that  $\deg L_2 = \alpha$ . Since  $L_1$  and  $L_2$  are homologous, it follows that there

is a (not framed) cobordism  $L$  between them. By definition, put  $h(L_2) = \bar{e}(L) \bmod 2d(\alpha)$ . (One can see that this is the Hopf invariant if  $\alpha = 0$  and  $L_1$  is null framed cobordant.)

It will follow from (1) and (2) below that  $h$  is well-defined:

- (1)  $h(L_2)$  does not depend on the choice of  $L$ ; and
- (2) if  $L_2$  and  $L'_2$  are framed cobordant then  $h(L_2) = h(L'_2)$ .

Let us first prove (2). Assume that  $L_1 \subset M \times 1$ ,  $L_2 \subset M \times 0$ ,  $L'_2 \subset M \times (-1)$ ,  $L \subset M \times [0, 1]$ . Let  $L' \subset M \times [-1, 0]$  be a framed cobordism between  $L_2$  and  $L'_2$ . By the geometric definition of the relative normal Euler class it follows that  $\bar{e}(L \cup L') = \bar{e}(L) + \bar{e}(L')$ . Since the cobordism  $L'$  is framed, it follows that  $\bar{e}(L') = 0$ . Thus  $\bar{e}(L \cup L') = \bar{e}(L)$ , and we obtain the required equality  $h(L_2) = h(L'_2)$ .

Let us prove (1). Take another general position cobordism  $L'$  between  $L_1$  and  $L_2$ . Assume that  $L_2 \subset M \times 0$ , two copies of  $L_1$  are contained in  $M \times (\pm 1)$  and  $L, L' \subset M \times [0, 1]$ . Let  $-L' \subset M \times [-1, 0]$  be the cobordism symmetric to  $L'$  (we consider the symmetry  $x \times t \rightarrow x \times (-t)$  on  $M \times \mathbb{R}$ ). Take a general position framed circle  $-L'_1 \subset M$  such that  $L_1 \cup L'_1$  is framed cobordant to zero, i. e. to an empty submanifold. Denote by  $\Delta$  the corresponding framed cobordism. Assume that two copies of  $L'_1$  are contained in  $M \times (\pm 1)$ , and  $\Delta \subset [1; +\infty)$ . Let  $-\Delta \subset (-\infty, -1]$  be the cobordism, symmetric to  $\Delta$ . Denote by

$$K = (-L') \cup L \cup \Delta \cup (L'_1 \times [-1, 1]) \cup (-\Delta).$$

By the geometric definition of the relative normal Euler class we obtain

$$\bar{e}(K) = \bar{e}(-L') + \bar{e}(L) + \bar{e}(\Delta) + \bar{e}(L'_1 \times [-1, 1]) + \bar{e}(-\Delta).$$

Here  $\Delta$ ,  $-\Delta$  and  $L'_1 \times [-1, 1]$  can be framed, so  $\bar{e}(\Delta) = \bar{e}(-\Delta) = \bar{e}(L'_1 \times [-1, 1]) = 0$ . Since the symmetry  $x \times t \rightarrow x \times (-t)$  reverses the orientation of  $M \times [-1, 1]$ , it follows by the geometric definition of the relative normal Euler class that  $\bar{e}(-L') = -\bar{e}(L')$ . Thus  $\bar{e}(K) = \bar{e}(L) - \bar{e}(L')$ . Now (1) follows from

$$\bar{e}(K) = -2\sigma = 2(-L' \cup L) \cap (L'_1 \times [-1, 1]) = 2K \cap (L''_1 \times \mathbb{R}) = 2[pK] \cdot \alpha = 0 \bmod 2d(\alpha).$$

Here  $\sigma$  is the difference between the numbers of positive and negative self-intersections of  $K$ , and the first equality follows from Lemma 3. The second equality follows from the construction of  $K$ . Then,  $L''_1 \subset M$  is a general position circle close to  $L'_1$  and homologous to it. By general position  $L'_1$  and  $L''_1$  are disjoint, so  $(-L' \cup L) \cap (L'_1 \times [-1, 1]) = K \cap (L''_1 \times [-1, 1])$ .

Since  $-\Delta$  is obtained from  $\Delta$  by the symmetry  $x \times t \rightarrow x \times (-t)$ , it follows that  $K \cap (L''_1 \times [1, +\infty)) = -K \cap (L''_1 \times (-\infty, -1])$ , and the third equality follows. Denote by  $p : M \times I \rightarrow M$  the projection. Then by general position we obtain the fourth equality, because the homological class of  $L''_1$  is  $\alpha$ . The last equality follows from the definition of  $d(\alpha)$ . So the proof of (1) is completed.

**Injectivity of  $h$ .** Let  $L_2$  and  $L'_2$  be a pair of framed 1-submanifolds such that  $h(L_2) = h(L'_2)$ . Let us prove that  $L_2$  and  $L'_2$  are framed cobordant. Assume that  $L_2 \subset M \times 1$ ,  $L_1 \subset M \times 0$  and  $L'_2 \subset M \times (-1)$ . Let  $L \subset M \times [0, 1]$  and  $-L' \subset M \times [-1, 0]$  be the cobordisms between  $L_1$  and  $L_2$ ,  $L_1$  and  $L'_2$  respectively. Since  $h(L_2) = h(L'_2)$ , it follows that  $\bar{e}(L) = -\bar{e}(-L') \bmod 2d(\alpha)$ . Then  $\bar{e}(-L' \cup L) = 2d(\alpha)y$  for some  $y \in \mathbb{Z}$ .

By the Poincaré duality there exists an element  $\beta \in H_2(M; \mathbb{Z})$  such that  $\alpha \cap \beta = d(\alpha)$ . Let  $K \subset M \times 0$  be a general position connected submanifold realizing the

class  $y\beta$ . Notice that  $\bar{e}(K) = 0$  by Lemma 3. Denote by  $K'$  the connected sum of  $(-L' \cup L)$  and  $K$  in  $M \times [-1; 1]$ . By the geometric definition of the relative normal Euler class it follows that  $\bar{e}(K') = \bar{e}(-L' \cup L) + \bar{e}(K) = 2d(\alpha)y$ .

The difference between the numbers of positive and negative self-intersection points of the manifold  $K'$  is equal to  $y\beta \cap \alpha = d(\alpha)y$ . Let  $K''$  be a new cobordism between  $L_2$  and  $L'_2$  obtained from  $K'$  by elimination of the self-intersection points. (Here we use a move in a neighbourhood of each self-intersection point analogous to the move taking the pair of the planes  $x = 0$ ,  $y = 0$  and  $z = 0$ ,  $t = 0$  to the surface

$$\begin{cases} x(\tau, \varphi) = \tau \cos \varphi, \\ y(\tau, \varphi) = \tau \sin \varphi, \\ z(\tau, \varphi) = (1 - \tau) \cos \varphi, \\ t(\tau, \varphi) = (1 - \tau) \sin \varphi; \end{cases}$$

in  $\mathbb{R}^4$  with coordinates  $(x, y, z, t)$ .) By the geometric definition of the normal Euler class it can be proved easily that removing of each self-intersection point decreases  $\bar{e}(K')$  by  $\pm 2$ , depending on the sign of the point (since our move is local, it suffices to prove it for a closed submanifold  $K'$ , and this latter case follows from Lemma 3). So  $\bar{e}(K'') = \bar{e}(K') - 2d(\alpha)y = 0$ . Thus  $K''$  can be framed, hence  $L_2$  and  $L'_2$  are framed cobordant.

**Surjectivity of  $h$ .** Let us construct a sequence  $L_1, L_2, \dots, L_{2d(\alpha)}$  of framed 1-submanifolds such that for  $j = 1, \dots, 2d(\alpha)$  we have  $h(L_j) = j - 1$ . Fix a homeomorphism  $L_1 \cong S^1$ . Denote by  $f_1(x)$  the basis vector of the fixed framing of  $L_1$  at the point  $x \in S^1$ . Take a map  $\varphi : S^1 \rightarrow SO(2)$  realizing the generator  $\pi_1(SO(2)) \cong \mathbb{Z}$ . For  $j = 2, \dots, 2d(\alpha)$  define the framing  $f_j$  of  $L_1$  by the formula  $f_j(x) = \varphi^{j-1}(x)f_1(x)$ . Let  $L_j$  be the submanifold  $L_1$  with framing  $f_j$ . Without loss of generality we may assume that  $h(L_2) \geq 0$ .

Let us prove that then  $h(L_2) = 1$ . Then it can be shown analogously that  $h(L_j) = j - 1$ . Take  $L = L_1 \times I$ . It suffices to construct a general position normal vector field on  $L$  extending the first field of the framing of  $L_1$  and  $L_2$  with a unique singular point. The normal bundle to  $L$  in  $M \times \mathbb{R}$  is trivial. Identify this bundle with  $\mathbb{R} \times \mathbb{R} \times L$  and denote by  $p_1, p_2 : \mathbb{R} \times \mathbb{R} \times L \rightarrow \mathbb{R}$  the projections to the first and the second multiples respectively. Further denote by  $f_2$  the first vector field of the framing  $f_2$ . Clearly,  $p_1 f_2(x)$ , where  $x \in L_2$ , has exactly two zeros. Join them by an arc  $A \subset L$ .

Analogously, join by an arc  $B$  the pair of zeros of  $p_2 f_2(x)$ . Clearly, we can choose the arcs  $A$  and  $B$  intersecting transversally at a single point. Take a general position normal vector field  $F_1$  on  $L$  extending the fields  $p_1 f_2$ ,  $p_1 f_1$  and such that  $p_2 F_1 = 0$ ,  $p_1 F_1|_A = 0$ . Analogously, extend  $p_2 f_2$  and  $p_2 f_1$  to a normal vector field  $F_2$  such that  $p_1 F_2 = 0$ ,  $p_2 F_2|_B = 0$ . The sum  $F_1 + F_2$  with a single zero at the point  $A \cap B$  is the required vector field.  $\square$

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